Quantum scalar field in FRW Universe with constant electromagnetic background

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We discuss massive scalar field with conformal coupling in Friedmann-Robertson-Walker (FRW) Universe of special type with constant electromagnetic field. Treating an external gravitational-electromagnetic background exactly, at first time the proper-time representations for out-in, in-in, and out-out scalar Green functions are explicitly constructed as proper-time integrals over the corresponding (complex) contours. The vacuum-to-vacuum transition amplitudes and number of created particles are found and vacuum instability is discussed. The mean values of the current and energy-momentum tensor are evaluated, and different approximations for them are investigated. The back reaction of the particles created to the electromagnetic field is estimated in different regimes. The connection between proper-time method and effective action is outlined. The effective action in scalar QED in weakly-curved FRW Universe (De Sitter space) with weak constant electromagnetic field is found as derivative expansion over curvature and electromagnetic field

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strength. Possible further applications of the results are briefly mentioned.
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I. INTRODUCTION

It is quite well known fact that quantum field theory in an external background is, generally speaking, theory with unstable vacuum. The vacuum instability leads to many interesting features, among which particles creation from vacuum is one of the most beautiful non-perturbative phenomena in quantum field theory. Furthermore, in interacting theories the vacuum instability may lead to quantum processes which are prohibited if the vacuum is stable. One ought to say that all the above mentioned peculiarities can not be reveal in frames of the perturbation theory with regards to the external background, one has to treat it exactly. The latter has been realized long ago by Schwinger [1] on the example of quantum electrodynamics in the constant electric field. The particles creation in this case has been calculated explicitly.

In quantum field theory with unstable vacuum it is necessary to construct different kinds of Green functions (GF), e.g. besides the causal GF (out-in GF) one has to use so called in-in GF, out-out GF, and so on [2–4] (for a review and technical details see [5]). General methods of such GF construction in electromagnetic (EM) background have been developed in [3,4]. The possible generalization of the formalism to an external gravitational background has been given in ref. [6]. Since the paper [1] it is known that causal (out-in) GF may be presented as a proper-time integral over a real infinite contour. At the same time, in the instable vacuum the in-in and out-out GF differ from the causal one. One can show [7] that these functions may be presented by the same proper-time integral (with the same integrand) but over another contours in the complex proper-time plane. The complete set of GF mentioned is necessary for the construction of Furry picture in interacting theories, and even in non-interacting cases one has to use them to define, for example, the back reaction of particles created and to construct different kinds of effective actions.

It may be likely that early Universe (EU) is filled with some type of electromagnetic fields. For example, recently (see [8,9] and references therein) the possibility of existence and role of primordial magnetic fields in EU have been discussed. From another point the possibility of existence of electromagnetic field in the EU has been discussed long ago in [10,11]. It has been shown there that the presence of the electrical field in the EU significally increases the gravitational particle creation from the vacuum. In principle, this process may be considered as a source for the dominant part of the Universe mass.

Having in mind the above cosmological motivations it is getting interesting to study the quantum field theory in curved background with electromagnetic field (of special form to be able to solve the problem analytically). In the present paper we are going to consider a massive charged scalar field with conformal coupling in the expanding FRW Universe with the scale factor $\Omega(\eta)$ (in terms of the conformal time) $\Omega^2(\eta) = b^2\eta^2 + a^2$. Such a scale factor corresponds to the expanding radiation dominated FRW Universe. In terms of physical time t the corresponding metric may be written as following:

$$ds^2 = dt^2 - \Omega^2(t)(dx^2 + dy^2 + dz^2) \ , \tag{1}$$

where for small times $|t| \ll a^2/b$, $\Omega^2(t) \simeq a^2[1 + (bt/a^2)^2]$ and for large times $|t| \gg a^2/b$, $\Omega^2(t) \simeq 2b|t|$ (see [10]). Moreover, such FRW Universe will be filled by the constant electromagnetic field (the precise form of this field is given in the next section). Thus, we start from the charged scalar theory in above background. Making conformal transformation of scalar theory (which works generally speaking for an arbitrary dimensionality of spacetime if correspondent conformal coupling is chosen), we remain with the theory in flat background but with time-dependent mass. For generality, we consider such a theory in d-dimensions (what may by useful also for Kaluza-Klein type theories [12]). The embedding of such a theory to d-dimensional constant EM field will be considered. (Of course, our main physical interest will be related with $d = 4$ where the Maxwell theory is conformally invariant, and one can start from the FRW Universe (1) with constant EM field from the very beginning, before the conformal transformation). In addition, in $d = 4$ the conformal

coupling (i.e. the choice of scalar gravitational coupling $\xi = 1/6$) is UV fixed point of renormalization group (RG) [12,13] for a variety of interacting theories (for example, for charged self-interacting scalar theory it is IR stable fixed point of RG [13]). Hence, there appears the additional motivation to start from conformal coupling $\xi = 1/6$, as anyway at high energies (strong curvatures) [12,13] ξ tends to this value. Hence, in Sect. II we solve d-dimensional Klein-Gordon equation for the theory in the constant EM field and with time-dependent mass to get complete sets of solutions classified as particles and antiparticles at $t \to \pm\infty$. Using them we construct all necessary GF for the scalar field as Schwinger type proper-time integrals. All GF have the same integrand and differ by the contours of integration in the complex proper-time plane. As far as we know that it is first explicit example for proper-time representation for complete set of scalar GF in gravitational-electromagnetic background (for pure electromagnetic background it was calculated in [7] and in pure gravitational background, see for example [14]). Using the exact solutions obtained we discuss particles creation under the combined effect of an external gravitational and electromagnetic fields. By means of GF we discuss the back reaction of the particles created and the effective action. Besides, in the Sect. IV we apply renormalization group (RG) method for calculation of the effective action in constant curvature spacetime filled by the constant EM field. The typical example of such spacetime is de Sitter Universe (exponentially expanding one). As specific theory, we consider scalar quantum electrodynamics (QED) on above-described background. The RG improved effective action is calculated as the derivative expansion over the external fields (hence, in this Sect. we consider weakly-curved spacetime filled by weak EM fields). Such an effective action represents the extension of the well-known Schwinger effective Lagrangian for the case of curved spacetime.

In the Conclusion the summary of the results obtained is presented. Some further prospects and possible applications are also discussed.

II. EXACT SOLUTIONS AND GREEN FUNCTIONS IN THE EXTERNAL TIME-DEPENDENT BACKGROUND

In this section we will present exact solutions and GF of the charged scalar field in the external constant uniform electromagnetic background. In addition, the former field will be considered in the time-dependent mass-like potential, which effectively reproduces effects of a gravitational background.

The scalar field obeys the Klein-Gordon equation:

$$\left(\hat{P}^2 - M^2\Omega^2(\eta)\right)\varphi = 0 \tag{2}$$

where $\hat{P}_\mu = i\partial_\mu - q\,A_\mu;\ q = -|e|$ (for electron); $x^0 \equiv \eta,\ \partial_0 \equiv \partial_\eta;\ \varphi(x)$ is the scalar field; the dimensionality of the space is chosen to be equal to $d = D+1,\ d \geq 2$ and the Minkowswski tensor has a form $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,\ldots)$. The time-dependent potential term is chosen as $\Omega^2(\eta) = b^2\eta^2 + a^2$. The external electromagnetic field will be chosen as the following: constant uniform electric field

$$F_{0D} = E\ \left(F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu A_\mu\right)\ , \tag{3}$$

and magnetic field (for $d > 3$), which is defined by $[d/2]-1$ invariants of the Lorentz transformations. This magnetic field is given by the corresponding components H_j (see for details [18]):

$$F^{(H)}_{\mu\nu} = \sum_{j=1}^{[d/2]-1} H_j\left(\delta^{2j}_\mu\delta^{2j-1}_\nu - \delta^{2j}_\nu\delta^{2j-1}_\mu\right)\ . \tag{4}$$

For such a field we select the following potentials:

$$A_0 = 0,\ A_D = Ex^0,\ A_i = A^\perp_i = -H_j\,x_{i+1}\,\delta_{i,2j-1},\ j = 1,\ldots,[d/2]-1,\ i = 1,\ldots,D-1. \tag{5}$$

In this case solutions φ of the equation (2) are related with ones φ' with some other choice of the potentials A'_μ for the same electromagnetic field, by the relation

$$\varphi' = \exp\left(iq\int_{x_c}^{x}\left(A_\mu - A'_\mu\right)dx^\mu\right)\varphi\ , \tag{6}$$

where the integral is taken along the line: $\partial_\nu \int_{x_c}^{x} \left(A_\mu - A'_\mu \right) dx^\mu = A_\nu - A'_\nu$.

Let us discuss first the physical motivations behind the Eq. (2). For $b^2 = 0$, $a^2 = 1$ we have d-dimensional scalar field in the constant electromagnetic field. If in this case $d > 4$ we have typical Kaluza-Klein type theory in external EM background (for review of quantum Kaluza-Klein theories, see, for example [12]).

The case $d = 4$, and $b^2 \neq 0$ is of special interest for us. In this case one can consider charged scalar field in the conformally-flat Universe (with scale factor $\Omega(\eta) = \sqrt{b^2\eta^2 + a^2}$) filled by the constant EM field. Choosing the scalar-gravitational coupling constant $\xi = 1/6$ (conformal coupling) and making the standard conformal transformation of the gravitational metric and scalar field, we come to the theory in flat spacetime with time-dependent mass. The corresponding field equation is given by (2) (EM field should not be transformed under the conformal transformation). Note that $\xi = 1/6$ is UV fixed point of RG for a variety of theories [13,12], so such conformal transformation may be used also for interacting theories (let us remind that $(\varphi^+\varphi)^2$-term is conformally invariant in the Lagrangian). Hence, $d = 4$ case in Eq. (2) actually corresponds to the quantum scalar field in the expanding FRW-Universe with constant EM field. However, for generality we leave d to be an arbitrary integer number.

As was already said, when quantum fields are considered in time-dependent backgrounds (electromagnetic or gravitational ones) one has to construct different Green functions. To this end one has to find special sets of exact solutions of the equation (2). Here we are going to describe such solutions. The functions $\varphi(x)$ can be presented in form:

$$\varphi_{p_D n}(x) = \varphi_{p_D n}(x_\parallel)\, \varphi_n(x_\perp) \quad , \tag{7}$$

where nonzero $x_\perp^i = x^i, \quad i = 1, \ldots, D-1, \quad x_\parallel^\mu = x^\mu,\ \mu = 0, D$ and

$$\hat{\mathbf{P}}_\perp^2 \varphi_n(x_\perp) = \mathbf{P}_\perp^2 \varphi_n(x_\perp), \quad P_\perp^i = P^i, \quad i = 1, \ldots, D-1\ , \tag{8}$$

$$\int \varphi_n^*(x_\perp)\varphi_{n'}(x_\perp) d\mathbf{x}_\perp = \delta_{nn'}\ . \tag{9}$$

Here $n = \left(n_1, \ldots, n_{[d/2]-1}, p_1, p_3, \ldots, p_{2[(d-1)/2]-1}\right)$ is complete set of the quantum numbers in the space $x_\perp$. The dimensionality of n is equal to $d-2$. The expression $\mathbf{P}_\perp^2$ is defined as

$$\mathbf{P}_\perp^2 = \sum_{j=1}^{[d/2]-1} \omega_j + \omega_0, \quad \omega_0 = \begin{cases} 0, & \text{d is even} \\ p_{d-2}^2, & \text{d is odd} \,, \end{cases} \tag{10}$$

$$\omega_j = \begin{cases} |qH_j|(2n_j+1),\ n_j = 0,1,\ldots \ , & H_j \neq 0 \\ p_{2j-1}^2 + p_{2j}^2 \ , & H_j = 0 \end{cases} ,$$

where in the presence of the magnetic field some momenta p_{2j} have to be replaced by the discrete quantum numbers n_j. The number of these momenta p_{2j} corresponds to the number of nonzero parameters H_j. Note that n includes the discrete n_j and continuous p_{2j-1}:

$$\left(\hat{P}_{2j-1}^2 + \hat{P}_{2j}^2\right) \varphi_n(x_\perp) = \omega_j \varphi_n(x_\perp) \,,$$

$$\hat{P}_{2j-1}\, \varphi_n(x_\perp) = p_{2j-1}\, \varphi_n(x_\perp) \,. \tag{11}$$

Let us write

$$\varphi_{p_D n}(x_\parallel) = \frac{1}{\sqrt{2\pi}}\, e^{-ip_D x^D} \varphi_{p_D n}(x^0) \quad , \tag{12}$$

where

$$\varphi_{p_D n}(x^0) = \varphi_{p_D n}(x^0, p_z)|_{p_z=0} \,,$$

and $\varphi(x^0, p_z)$ is a solution of equation

$$\left[\left(i\frac{\partial}{\partial \widetilde{\eta}}\right)^2 - (p_z - \rho\widetilde{\eta})^2 - \rho\lambda\right] \varphi_{p_D n}\left(x^0, p_z\right) = 0 \,, \tag{13}$$

with

$$\widetilde{\eta} = x^0 - \frac{1}{\rho^2}\, qEp_D,\ \rho^2 = (qE)^2 + (bM)^2, \quad \rho\lambda = (aM)^2 + p_D^2 \frac{(bM)^2}{\rho^2} + \mathbf{P}_\perp^2 \,.$$

Orthonormalized and classified as particles (+) and antiparticles (-) at $x^0 \to \pm\infty$ solutions of the equation (13) have the form [19]

$$
{}_{+}^{-}\varphi_{p_D n}\left(x^0, p_z\right) = BD_\nu[\pm(1-i)\xi], \;\; {}_{-}^{+}\varphi_{p_D n}\left(x^0, p_z\right) = BD_{-\nu-1}\left[\pm(1+i)\xi\right] \;,
$$
$$
\xi = \frac{1}{\sqrt{\rho}}\left(\rho\tilde{\eta} - p_z\right), \; \nu = \frac{i\lambda}{2} - \frac{1}{2}, \quad B = (2\rho)^{-1/4}\exp\left(-\frac{\pi\lambda}{8}\right), \tag{14}
$$

see [18] for additional arguments advocating such a classification.

One can find decomposition coefficients $G\left({}_\zeta|^{\zeta'}\right)$ of the out-solutions in the in-solutions,

$$
{}^\zeta\varphi(x) = {}_+\varphi(x)G\left({}_+|^\zeta\right) - {}_-\varphi(x)G\left({}_-|^\zeta\right)\;, \;\; \zeta = \pm\;. \tag{15}
$$

The matrices $G\left({}_\zeta|^{\zeta'}\right)$ obey the following relations,

$$
\begin{aligned}
&G\left({}_\zeta|^+\right)G\left({}_\zeta|^+\right)^\dagger - G\left({}_\zeta|^-\right)G\left({}_\zeta|^-\right)^\dagger = \zeta I,\\
&G\left({}_+|^+\right)G\left({}_-|^+\right)^\dagger - G\left({}_+|^-\right)G\left({}_-|^-\right)^\dagger = 0\;.
\end{aligned} \tag{16}
$$

The latter can be derived from the orthonormality conditions. One can easily see that $G\left({}_\zeta|^{\zeta'}\right)$ are diagonal,

$$
G\left({}_\zeta|^{\zeta'}\right)_{ll'} = \delta_{l,l'}\; g\left({}_\zeta|^{\zeta'}\right)\;, \;\; l = (p_D, n)\;, l' = (p'_D, n')\;, \tag{17}
$$

where

$$
g({}_\zeta|^{\zeta'}) = {}_\zeta\varphi^*_{p_D n}(x^0, p_z)\, i\, \overleftrightarrow{\partial}_0\; {}^{\zeta'}\varphi_{p_D n}(x^0, p_z)\;. \tag{18}
$$

As was already said, in case of theories with unstable vacuum one has to study different types of GF, which we define via $\Delta,\ \Delta^c,\ \Delta^\mp,\ \Delta^c_{in},\ \Delta^\mp_{in}$, and so on, according [3–5]. For example, the field theoretical definitions of causal out-in, in-in and out-out GF have the form

$$
\Delta^c(x,x') = c_v^{-1} i < 0, out|T\phi(x)\phi^\dagger(x')|0, in >, \;\; c_v =< 0, out|0, in >\;, \tag{19}
$$

$$
\Delta^c_{in}(x,x') = i < 0, in|T\phi(x)\phi^\dagger(x')|0, in >\;, \tag{20}
$$

$$
\Delta^c_{out}(x,x') = i < 0, out|T\phi(x)\phi^\dagger(x')|0, out >\;, \tag{21}
$$

where $\phi(x)$ is a quantized scalar field, $|0, in>$ and $|0, out>$ are initial and final vacua and c_v is vacuum to vacuum transition amplitude. In principle, one can calculate all of GF using sets of solutions (7) and some relations between the functions,

$$\Delta^c(x,x') = \theta\left(x_0 - x'_0\right)\,\Delta^-(x,x') - \theta\left(x'_0 - x_0\right)\Delta^+(x,x') \quad , \tag{22}$$

$$\Delta(x,x') = i\left[\phi(x),\phi^\dagger(x')\right]_- = \Delta^-(x,x') + \Delta^+(x,x') \quad , \tag{23}$$

$$\Delta^-(x,x') = i\int_{-\infty}^{+\infty} dp_D \sum_n {}^+\varphi_{p_D n}(x) g({}_+|{}^+)^{-1}{}_+\varphi^*_{p_D n}(x') \quad ,$$

$$\Delta^+(x,x') = -i\int_{-\infty}^{+\infty} dp_D \sum_n {}_-\varphi_{p_D n}(x)\left[g({}_-|{}^-)^{-1}\right]^*{}^-\varphi^*_{p_D n}(x') \quad , \tag{24}$$

$$\Delta^c{}_{in}(x,x') = \theta(x_0 - x'_0)\Delta^-_{in}(x,x') - \theta(x'_0 - x_0)\Delta^+{}_{in}(x,x') \quad ,$$

$$\Delta^\mp_{in}(x,x') = \pm i\int_{-\infty}^{+\infty} dp_D \sum_n {}_\pm\varphi_{p_D n}(x)\,{}_\pm\varphi^*_{p_D n}(x') \quad , \tag{25}$$

$$\Delta^c{}_{out}(x,x') = \theta(x_0 - x'_0)\Delta^-_{out}(x,x') - \theta(x'_0 - x_0)\Delta^+{}_{out}(x,x') \quad ,$$

$$\Delta^\mp_{out}(x,x') = \pm i\int_{-\infty}^{+\infty} dp_D \sum_n {}^\pm\varphi_{p_D n}(x)\,{}^\pm\varphi^*_{p_D n}(x') \quad . \tag{26}$$

Below we are going to do such calculations in the case under consideration. First, one can remark that the functions $\Delta^\mp$ and $\Delta^\mp_{in}$ can be presented as follows

$$\pm\Delta^\mp(x,x') = \Delta^c(x,x') \pm \theta(\mp(x_0 - x'_0))\Delta(x,x') \quad , \tag{27}$$

$$\pm\Delta^\mp_{in}(x,x') = \Delta^c_{in}(x,x') \pm \theta(\mp(x_0 - x'_0))\Delta(x,x') \quad , \tag{28}$$

$$\pm\Delta^\mp_{out}(x,x') = \Delta^c_{out}(x,x') \pm \theta(\mp(x_0 - x'_0))\Delta(x,x') \quad , \tag{29}$$

$$\Delta^c_{in}(x,x') = \Delta^c(x,x') - \Delta^a(x,x') \quad , \tag{30}$$

$$\Delta^c_{out}(x,x') = \Delta^c(x,x') - \Delta^p(x,x') \quad , \tag{31}$$

$$\Delta^a(x,x') = -i\int_{-\infty}^{+\infty} dp_D \sum_n {}_-\varphi_{p_D n}(x)\,\left[g({}_-|{}^-)^{-1}g({}_+|{}^-)\right]^*{}_+\varphi^*_{p_D n}(x') \quad , \tag{32}$$

$$\Delta^p(x,x') = -i\int_{-\infty}^{+\infty} dp_D \sum_n {}^+\varphi_{p_D n}(x)\,g({}_+|{}^+)^{-1}g({}_+|{}^-){}^-\varphi^*_{p_D n}(x') \quad . \tag{33}$$

The coefficients (18) do not depend on p_z, thus, one can present the functions $\Delta^\mp$ and $\Delta^{a,p}$ in the convenient form

$$\Delta^{\mp,a,p}(x,x') = \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} \frac{dp_D}{2\pi}\, e^{-ip_D y^D}\,\Delta^{\mp,a,p}_Q \;, \quad y_\mu = x_\mu - x'_\mu \;, \tag{34}$$

$$\Delta_Q^{\mp,a,p} = \Delta_Q^{\mp,a,p}\left(\tilde{\eta}, x_\perp, \tilde{\eta}', x'_\perp, z - z', p_D\right) \ ,$$

$$\Delta_Q^{-} = i\sum_n \int_{-\infty}^{+\infty} dp_z\, {}^{+}\varphi_{p_Dn}(\tilde{\eta}, x_\perp, z, p_z) g({}_{+}|^{+})^{-1}\, {}_{+}\varphi^*_{p_Dn}\left(\tilde{\eta}', x'_\perp, z', p_z\right) \quad ,$$

$$\Delta_Q^{+} = -i\sum_n \int_{-\infty}^{+\infty} dp_z\, {}_{-}\varphi_{p_Dn}(\tilde{\eta}, x_\perp, z, p_z) \left[g({}_{-}|^{-})^{-1}\right]^*\, {}^{-}\varphi^*_{p_Dn}\left(\tilde{\eta}', x'_\perp, z', p_z\right) \quad ,$$

$$\Delta_Q^{a} = -i\sum_n \int_{-\infty}^{+\infty} dp_z\, {}_{-}\varphi_{p_Dn}(\tilde{\eta}, x_\perp, z, p_z) \left[g({}_{-}|^{-})^{-1} g({}_{+}|^{-})\right]^*\, {}_{+}\varphi^*_{p_Dn}\left(\tilde{\eta}', x'_\perp, z', p_z\right) \quad , \quad (35)$$

$$\Delta_Q^{p} = -i\sum_n \int_{-\infty}^{+\infty} dp_z\, {}^{+}\varphi_{p_Dn}(\tilde{\eta}, x_\perp, z, p_z) g({}_{+}|^{+})^{-1} g({}_{+}|^{-})\, {}^{-}\varphi^*_{p_Dn}\left(\tilde{\eta}', x'_\perp, z', p_z\right) \quad , \quad (36)$$

where

$$\varphi_{p_Dn}\left(\tilde{\eta}, x_\perp, z, p_z\right) = \varphi_{p_Dn}(\tilde{\eta}, z, p_z)\varphi_n(x_\perp) \quad ,$$

$$\varphi_{p_Dn}\left(\tilde{\eta}, z, p_z\right) = \frac{1}{\sqrt{2\pi}}\, e^{-ip_z z}\varphi_{p_Dn}(x^0, p_z) \quad .$$

The functions $\Delta_Q^{\mp,a,p}$ have the form of the corresponding GF in EM background, where $\tilde{\eta}$ is the time, z is the coordinate along the electric field, the mass $m_Q^2 = a^2M^2 + p_D^2\frac{(bM)^2}{\rho^2}$, and the potential of the electromagnetic field is $A_z = \tilde{\eta}\frac{\rho}{q}$. The functions

$$\psi(\tilde{\eta}, z) = \int_{-\infty}^{+\infty} \varphi_{p_Dn}\left(\tilde{\eta}, z, p_z\right) c_{np_z}\, dp_z \quad (37)$$

with some coefficient c_{np_z}, obey the equation

$$\left(\left(i\partial_{\tilde{\eta}}\right)^2 - \left(i\partial_z - \rho\tilde{\eta}\right)^2 - \mathbf{P}_\perp^2 - m_Q^2\right)\varphi_Q(\tilde{\eta}, z) = 0 \quad . \quad (38)$$

Its solutions can be found [20] in the form of decompositions in full sets of solutions of some first order equations in the light-cone variables, $z_\mp = \tilde{\eta} \mp z$,

$${}^{-}_{+}\varphi(\tilde{\eta}, z, p_-) = \frac{1}{\sqrt{4\pi}}\,\rho^{-1/4}\,\exp\left\{\frac{i\rho}{2}\left(\frac{1}{2}z_-^2 - z^2\right) - i\,\frac{p_-}{2}\,z_+ + \nu\ln(\mp\tilde{\pi}_-)\right\},$$

$$\pi_- = \sqrt{\rho}\,\tilde{\pi}_- = p_- - \rho z_- \quad , \quad \ln(\mp\tilde{\pi}_-) = \ln(\tilde{\pi}_-) + i\pi\theta(\pm\tilde{\pi}_-) \quad ,$$

$${}^{+}\varphi(\tilde{\eta}, z, p_-) = \theta(\tilde{\pi}_-)\, {}_{+}\varphi(\tilde{\eta}, z, p_-) g({}_{+}|^{+}) \quad ,$$

$${}_{-}\varphi(\tilde{\eta}, z, p_-) = \theta(-\tilde{\pi}_-)\, {}^{-}\varphi(\tilde{\eta}, z, p_-) g({}_{-}|^{-})^* \quad , \quad (39)$$

and $\pm$ classification has the same meaning as one in (14). Such decompositions have the form of integrals over the parameters p_-. Then one can present the integrals (35) by means of those p_- integrals,

$$\mp\Delta_Q^{\pm} = \int_{-\infty}^{+\infty} \theta(\mp\tilde{\pi}_-)\,{}^{-}_{+}\tilde{f}dp_- \ , \tag{40}$$
$${}^{-}_{+}\tilde{f} = i\sum_n {}^{-}_{+}\varphi(\tilde{\eta}, z, p_-)\varphi_n(x_\perp){}^{-}_{+}\varphi^*(\tilde{\eta}', z', p_-)\varphi_n^*(x'_\perp) \ .$$

Taking into account that $|g({}_-|{}^+)|^2 = e^{-\pi\lambda}$, one gets

$$\Delta_Q^a = -\int_{-\infty}^{+\infty} \theta(-\tilde{\pi}_-)\,{}_+\tilde{f}dp_- \ ,$$
$$\Delta_Q^p = -\int_{-\infty}^{+\infty} \theta(\tilde{\pi}_-)\,{}^-\tilde{f}dp_- \ . \tag{41}$$

It was shown in [7] that the p_- integrals can always be transformed into integrals over the proper-time s. After the summation over n, which can be done similarly to the case $d = 4$ [7], one gets a result which can be written as

$$\pm\Delta_Q^{\mp} = \Delta_Q^c \pm \theta(\mp y_0)\Delta_Q \quad , \tag{42}$$

$$\Delta_Q^c = \int_{\Gamma_c} f_Q ds \quad , \ f_Q = f_Q\left(\tilde{\eta}, x_\perp, \tilde{\eta}', x'_\perp, z - z', p_D, s\right) \ , \tag{43}$$

$$\Delta_Q = \epsilon(y_0) \int_{\Gamma_c - \Gamma_2 - \Gamma_1} f_Q ds \quad , \quad \epsilon(y_0) = \mathrm{sgn}(y_0) \quad , \tag{44}$$

$$\Delta_Q^a = \int_{\Gamma_a} f_Q ds + \theta(z' - z) \int_{\Gamma_3 + \Gamma_2 - \Gamma_a} f_Q ds \ , \tag{45}$$
$$\Delta_Q^p = \int_{\Gamma_a} f_Q ds + \theta(z - z') \int_{\Gamma_3 + \Gamma_2 - \Gamma_a} f_Q ds \ , \tag{46}$$

where $\theta(0) = 1/2$.

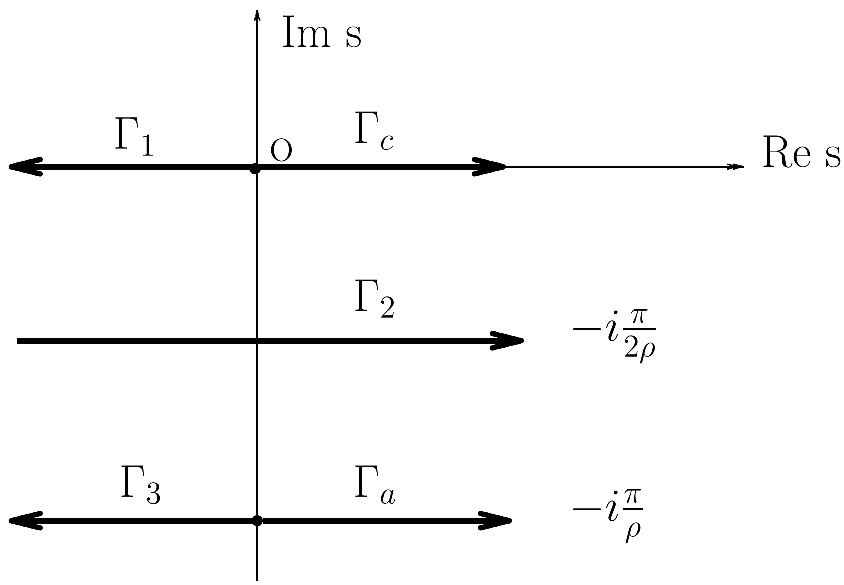

FIG. 1. Contours of integration $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_c, \Gamma_a$

$$f_Q = \exp\{-iq\int_{x'}^{x} A_\mu^\perp dx^\mu\} f_Q^\parallel\left(\widetilde{\eta}, \widetilde{\eta}', z - z', p_D, s\right) f_\perp\left(x_\perp, x'_\perp, s\right) \quad , \tag{47}$$

$$f_\perp(y_\perp, s) = c_d \prod_{j=1}^{(d-2)/2}\left(\frac{qH_j}{\sin(qH_j s)}\right), \qquad d \text{ is even,}$$

$$f_\perp(y_\perp, s) = c_d s^{-1/2}\prod_{j=1}^{(d-3)/2}\left(\frac{qH_j}{\sin(qH_j s)}\right), \quad d \text{ is odd,} \tag{48}$$

$$c_d = (4\pi)^{-d/2}\exp\left\{-\frac{i}{4}\left[\pi(d-4) + y_\perp qF^{(H)}\coth(qF^{(H)}s)y_\perp\right]\right\} ,$$

$$f_Q^\parallel\left(\widetilde{\eta}, \widetilde{\eta}', z, p_D, s\right) = \frac{\rho}{\sinh(\rho s)}\exp\left\{-i\frac{\rho}{2}(\widetilde{\eta} + \widetilde{\eta}')z - im_Q^2 s\right.$$

$$\left. + i\frac{\rho}{4}\left[z^2 - (\widetilde{\eta} - \widetilde{\eta}')^2\right]\coth(\rho s)\right\} \quad , \tag{49}$$

whereas the following relations take place

$$i\frac{d}{ds}f_Q = \left(m_Q^2 - P_Q^2\right) f_Q \quad , \tag{50}$$
$$\hat{P}_Q^2 = (i\partial_{\widetilde{\eta}})^2 - (i\partial_z - \rho\widetilde{\eta})^2 - \hat{\mathbf{P}}_\perp^2 \quad ,$$

$$\lim_{s\to+0} f_Q = i\delta(\widetilde{\eta} - \widetilde{\eta}')\,\delta(z - z')\delta(y_\perp) \quad . \tag{51}$$

In accordance with (34) one can calculate the Gaussian integrals over p_D and z for all the points s on the contours Fig.1. As a result one gets

$$\Delta^c(x,x') = \int_{\Gamma_c} f(x,x',s)ds \quad , \tag{52}$$

$$\Delta(x,x') = \epsilon(y_0) \int_{\Gamma} f(x,x',s)ds \quad , \tag{53}$$

$$\Delta^a(x,x') = -\Delta^{(1)}(x,x') - \Delta^{(2)}(x,x') \quad , \tag{54}$$

$$\Delta^p(x,x') = -\Delta^{(1)}(x,x') + \Delta^{(2)}(x,x') \quad , \tag{55}$$

$$\Delta^{(1)}(x,x') = -\frac{1}{2} \int_{\Gamma_3+\Gamma_2+\Gamma_a} f(x,x',s)ds \quad , \tag{56}$$

$$\Delta^{(2)}(x,x') = \int_{\Gamma_3+\Gamma_2-\Gamma_a} f_r(x,x',s)ds \quad , \tag{57}$$

where

$$f_r(x,x',s) = \frac{1}{2\sqrt{\pi}} \gamma\left(\frac{1}{2},\alpha\right) f(x,x',s) \quad ,$$

$$\alpha = e^{-i\pi/2} \frac{1}{4s(bM)^2\omega} \left[(x_0 + {x'}_0)\, s(bM)^2 + qEy^D \right]^2 \quad , \tag{58}$$

and $\gamma\left(\frac{1}{2},\alpha\right)$ is the incomplete gamma-function. Here

$$f(x,x',s) = e^{iq\Lambda} f_{\parallel}\left(x_0, {x'}_0, y^D, s\right) f_{\perp}(y_{\perp}, s) \quad , \tag{59}$$

$$f_{\parallel}\left(x_0, {x'}_0, y^D, s\right) = \frac{\rho}{\sinh(\rho s)\omega^{1/2}} \exp\left\{ i\frac{qE}{2}(x_0 + {x'}_0)\, y^D - i\frac{\rho}{4}(x_0 - {x'}_0)^2 \coth(\rho s) \right.$$

$$\left. -i(aM)^2 s + i\frac{\rho}{4\omega} y_D^2 \coth(\rho s) - \frac{i}{4\omega}\left[(bM)^2 s\,(x_0 + {x'}_0)^2 + 2qEy^D (x_0 + {x'}_0) \right] \right\} \quad ,$$

$$\omega = \frac{(bM)^2}{\rho} s \coth(\rho s) + \frac{(qE)^2}{\rho^2} \quad . \tag{60}$$

Here only Λ depends on the choice of the gauge for the constant field, via the integral

$$\Lambda = -\int_{x'}^{x} A_\mu dx^\mu \quad , \tag{61}$$

which is taken along the line. To get the function f in an arbitrary gauge A', one has only to replace A by A' in the Λ.

One can see that

$$i\frac{d}{ds} f(x,x',s) = \left(M^2\Omega^2(x^0) - \hat{P}^2 \right) f(x,x',s) \quad , \tag{62}$$

$$\lim_{s\to+0} f(x,x',s) = i\delta(x-x') \quad . \tag{63}$$

Thus, $f(x,x',s)$ is Fock-Schwinger function [1,21]. The contour $\Gamma_c - \Gamma_2 - \Gamma_1$ in (53) was transformed into Γ after the integration over p_D and z. Then the results are consistent with the general expression for the commutation function obtained in [22]. For appropriate choice of gauge and in $d = 4$ the function $f(x,x',s)$ coincides with one from [11].

If $b \neq 0$, then the function $f(x,x',s)$ has three singular points on the complex region $-\pi \leq \Im(\rho s) \leq 0$ which are distributed at the imaginary axis: $\rho s_0 = 0$, $\rho s_1 = -i\pi$ and $\rho s_2 = -ic_2$. The latter point is connected with zero value of the function ω. We get an equation for c_2 from the condition $\omega = 0$,

$$c_2 \tan(c_2 - \pi/2) - \left(\frac{qE}{bM}\right)^2 = 0 \, , \tag{64}$$

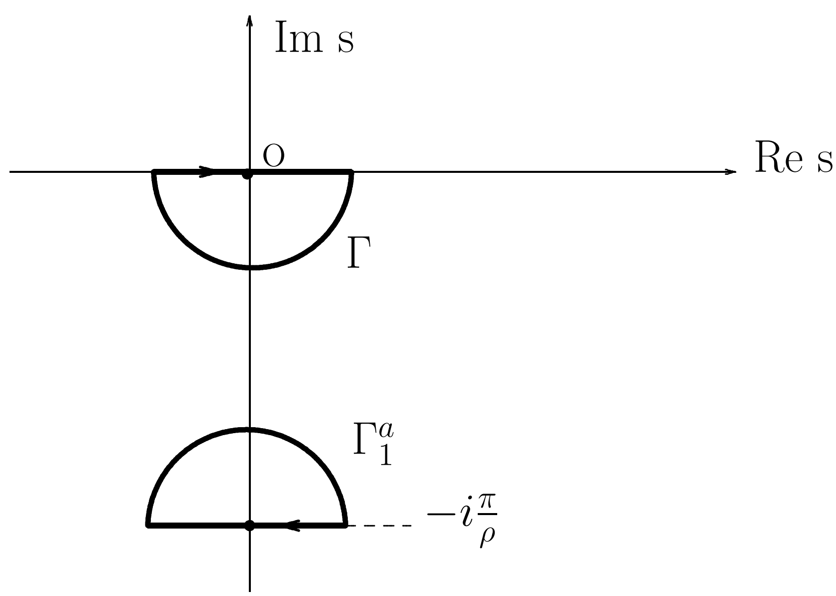

FIG. 2. Contours of integration Γ, Γ_1^a

where $\pi/2 < c_2 < \pi$. The position of this point depends on the ratio $qE/(bM)$, e.g. at $bM/(qE) \to 0$ one has $c_2 \to \pi$ and at $qE/(bM) \to 0$ one has $c_2 \to \pi/2$. Notice, in the case $E = 0$ it is convenient to put $c_2 = \pi/2 + 0$ because of the contour Γ_2 must be passed above the singular point s_2 in the case as well.

If $b = 0$, then $\omega = 1$ and the function $f(x,x',s)$ has only two singular points s_0 and s_1 on the above mentioned complex region. In this case the gauge invariant function $f_{\|}$ does not depend on on $x_0 + x_0'$. In this degenerate case it follows from (34), (45) and (46),

$$\Delta^a(x,x') = \int_{\Gamma_a} f(x,x',s)ds + \theta(-y^D)\int_{\Gamma_3+\Gamma_2-\Gamma_a} f(x,x',s)ds\ , \tag{65}$$

$$\Delta^p(x,x') = \int_{\Gamma_a} f(x,x',s)ds + \theta(y^D)\int_{\Gamma_3+\Gamma_2-\Gamma_a} f(x,x',s)ds\ . \tag{66}$$

Let us return to more interesting case $b \neq 0$. Our aim is to demonstrate that function $\Delta^{(2)}(x,x')$ from (57) can also be presented via a proper-time integral with the kernel $f(x,x',s)$ as it was done for all other Δ-functions. To this end let us transform the contour $\Gamma_3 + \Gamma_2 - \Gamma_a$ in (57) into two ones: Γ_1^a (see FIG.2) and $\Gamma_l + \Gamma_r$ (see FIG.3).

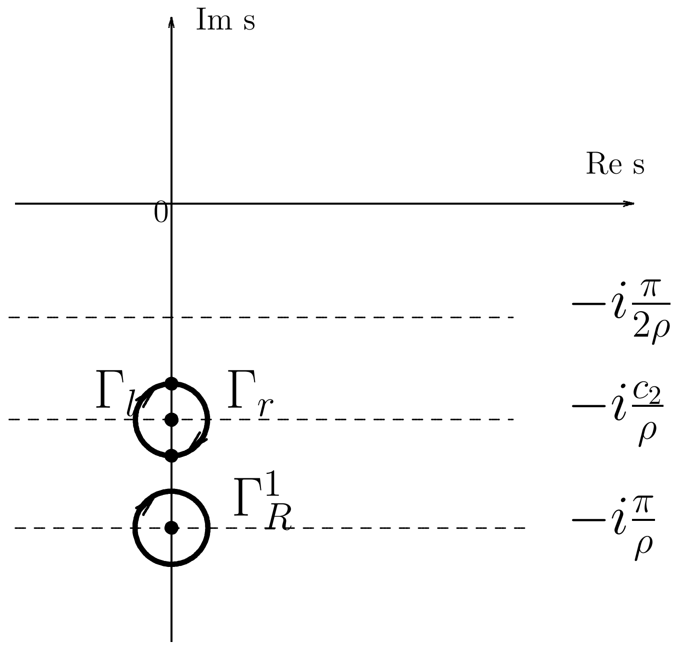

FIG. 3. Contours of integration Γ_R^1, Γ_l and Γ_r

The radius of the contour Γ_1^a tends to zero. The contour $\Gamma_l + \Gamma_r$ is a infinitesimal radius clockwise circle around the singular point s_2. However, it is convenient to present it as a combination of two semicircles Γ_l and Γ_r placed on the left and the right sides of the imaginary axis respectively. The argument $\arg s'$ of the Γ_l radius is in the interval $\pi/2 \leq \arg s' \leq 3\pi/2$ and of the Γ_r radius is in the interval $-\pi/2 \leq \arg s' < \pi/2$. Then (57) can be rewritten in the form

$$\Delta^{(2)}(x,x') = \int_{\Gamma_l+\Gamma_r} f_r(x,x',s)ds + r(x,x') \quad , \tag{67}$$

$$r(x,x') = \int_{\Gamma_1^a} f_r(x,x',s)ds\ . \tag{68}$$

Taking into account (B1) one gets

$$f_r\left(x, x', s' - i\frac{\pi}{\rho}\right) \underset{s'\to 0}{\longrightarrow} \mathrm{const}\cdot\exp\{-\frac{i}{4s'}\,(x_0 - x'_0)^2\} \quad .$$

Hence, one can see that $r(x, x') = 0$, $\partial_0 r(x, x') = 0$ at any $x_0 - x'_0$. Moreover, using (62), it is easy to see that the distribution $r(x, x')$ obeys the equation (2). Thus, $r(x, x')$ is equal to zero identically. The function $f_r(x, x's)$ is 2π periodic function of the argument $\arg s'$ of the Γ_l and Γ_r radiuses. One needs to take into account the asymptotic decomposition (A6) which is valid in the region $-3\pi/2 < \arg\alpha < 3\pi/2$. Then, using (A6) one gets from (67)

$$\Delta^{(2)}(x, x') = \frac{1}{2}\begin{cases} -\int_{\Gamma_l+\Gamma_r} f(x, x's)ds\ , & -5\pi/4 < \beta < -3\pi/4, \\ -\int_{\Gamma_l-\Gamma_r} f(x, x's)ds\ , & -3\pi/4 \le \beta < -\pi/4, \\ \int_{\Gamma_l+\Gamma_r} f(x, x's)ds\ , & -\pi/4 \le \beta \le \pi/4, \\ \int_{\Gamma_l-\Gamma_r} f(x, x's)ds\ , & \pi/4 < \beta \le 3\pi/4, \end{cases} \tag{69}$$

where $\beta = \arg\left[(x_0 + x'_0)c_2(bM)^2(-i) + \rho q E y^D\right]$.

One can verify that expression (69) is continuous in the boundaries of the β intervals. Then, using (62), one can demonstrate that the representation (69) obeys the equation (2). One can also verify that the representation $\Delta^{(1)}(x, x')$ (56) obeys the equation (2). Thus, all the Δ-Green functions considered here, excluding those marked by the index "c", are solutions of the equation (2). The important difference between basic Green functions $\Delta^c(x, x')$, $\Delta^{(1)}(x, x')$ and $\Delta(x, x')$, $\Delta^{(2)}(x, x')$ is that the first ones are symmetric under simultaneous change of sign in x_0, x'_0, x_D, x'_D and the seconds ones change sign in this case.

Note finally that using proper-time kernel $f(x, x', s)$ (59) one can easily construct Schwinger out-in effective action

$$\Gamma_{out-in} = -i\int dx \int_0^\infty s^{-1}\, f(x, x, s)\, ds; .$$

Similar out-in effective action (but in another approximation) will be discussed in Section IV for scalar electrodynamics. As regards to in-in effective action its representation is more complicated [4,5] and will be discussed in the next publication.

III. VACUUM INSTABILITY AND BACK REACTION OF PARTICLES CREATED

All the information about the processes of particles creation, annihilation, and scattering in an external field (without radiative corrections) can be extracted from the matrices $G\left({}_{\zeta}|{}^{\zeta'}\right)$ (15). These matrices define a canonical transformation between in and out creation and annihilation operators in the generalized Furry representation [3,5],

$$
\begin{aligned}
a^{\dagger}(out) &= a^{\dagger}(in)G\left({}_{+}|{}^{+}\right) + b(in)G\left({}_{-}|{}^{+}\right), \\
-b(out) &= a^{\dagger}(in)G\left({}_{+}|{}^{-}\right) + b(in)G\left({}_{-}|{}^{-}\right).
\end{aligned} \tag{70}
$$

Here $a_l^{\dagger}(in)$, $b_l^{\dagger}(in)$, $a_l(in)$, $b_l(in)$ are creation and annihilation operators of in-particles and antiparticles respectively and $a_l^{\dagger}(out)$,$b_l^{\dagger}(out)$, $a_l(out)$, $b_l(out)$ are ones of out-particles and antiparticles, l are possible quantum numbers (in our case $l = p_D, n$) . For example, the mean numbers of particles created (which are also equal to the numbers of pairs created) by the external field from the in-vacuum $|0, in>$ with a given quantum number l is

$$
N_l =< 0, in|a_l^{\dagger}(out)a_l(out)|0, in >= \left|g\left({}_{-}|{}^{+}\right)\right|^2 . \tag{71}
$$

(for a review of gravitational particles creation, see [24,29].) The standard space coordinate volume regularization was used to get the latter formula, so that $\delta(p_j - p'_j) \rightarrow \delta_{p_j,p'_j}$. The probability for a vacuum to remain a vacuum is

$$
P_v = |c_v|^2 = \exp\left\{-\sum_l \ln\left(1 + N_l\right)\right\} , \tag{72}
$$

where $|0, out>$ is the out-vacuum.

Remember that we are discussing the case in which the electric field acts for an infinite time. However, one can analyse the problem in finite times $T = x^0_{out} - x^0_{in}$, acting similar to [18]. Then the mean numbers of (p_D, n)- particles created by the external field are

$$
N_{p_D n} = |g(-|^{+})|^2 = e^{-\pi\lambda}, \text{ if } \quad \sqrt{\rho}\,T \gg 1, \text{ and } \sqrt{\rho}T \gg \lambda, \text{ and } \rho^2 T \gg |qEp_D| , \tag{73}
$$

where λ is defined in (13). The latter conditions take place for large T.

At d=4 (73) coincides with the one obtained in [10], and at $b = 0$ (the gravitational field is absent) it coincides with the one obtained in [18]. $(bM)^2/\rho^2 < 1$, and the number $N_{p_D n}$ depends from p_D^2 more weaker than from other quantum numbers $\mathbf{P}_\perp^2$.

If the condition $p_D^2(bM)^2/\rho^3 << 1$ takes place (the gravitational field is in a sense weaker than the electric one) the p_D dependence of the mean numbers (73) is similar to the pure electrodynamical case. Thus [18] one can evaluate that $\int dp_D = (eE)^{-1}\rho^2 T$. Then one can estimate the particle creation per unit of time similarly to [10]. In strong enough gravitational fields time dependence of the effect is nonlinear one and demands a special study.

To get the total number N of particles created one has to sum over the quantum numbers n, p_D. The sum over the momenta can be easily transformed into an integral. Thus, if $b = 0$ one gets result presented in [18]. If $b \neq 0$ and $d = 4$ the total number of pairs created per space coordinate volume has the form

$$\tilde{n} = \frac{\sum_{p_D,n} N_{p_D n}}{\int d\mathbf{x}} = \frac{\beta(1)}{8\pi^2}\frac{\rho^{3/2}}{bM}\exp\left\{-\pi\frac{(aM)^2}{\rho}\right\}, \tag{74}$$

where

$$\beta(n) = \frac{qH}{\sinh(n\pi qH/\rho)} \; .$$

The observable number density of the created pairs in the asymptotic region $x_0 = x_0^{out} \to \infty$ is given by

$$n^{cr} = \tilde{n}/\Omega^3(x_0) \; . \tag{75}$$

These results coincide with ones in [10].

In case $b \to 0$ the expression (74) is growing unlimited. In this case the particles are created in main by the electric field, whereas the parameter b plays a role of “cut-off” factor, which eliminates creation of particles with extremely high momenta along the electric field. One can see that from the expression (73). Thus, the limit $b \to 0$ corresponds to the case of the electric field which acts an infinite time when the number of particles created is proportional to the time of the field action. In fact, we are interesting in the

case when the field action time T is big enough to obey the stabilization condition, which has the form $T >> (qE)^{-1/2}$ for intense field. As was already remarked above, in this case $\int dp_D = (qE)^{-1}\rho^2 T$. Then it is clear that parameter b has to be interpreted as a quantity which is inversely proportional to the field action time T making the substitution $b^{-1} \to TM(\pi\sqrt{\rho})^{-1}$.

The vacuum-to-vacuum transition probability can be calculated, using formula (72). Thus, we get an analog of the well-known Schwinger formula [1] in the case under consideration. For the case $b = 0$ the result was given in [18]. For the case $b \neq 0$ and $d = 4$ one gets

$$P_v = \exp\left\{-\mu\tilde{n}\int d\mathbf{x}\right\}, \quad \mu = \sum_{l=0}^{\infty} \frac{(-1)^l \beta(l+1)}{(l+1)^{3/2}\beta(1)} \exp\left\{-l\pi\frac{(aM)^2}{\rho}\right\} . \tag{76}$$

This result coincides with the one obtained in [11].

Let operator of current of scalar field operator $\phi(x)$ obeying the Klein-Gordon equation (2) has a form

$$j_\mu = q\left(\hat{P}_\mu + \hat{P'}^*_\mu\right)\frac{1}{2}\left[\phi^\dagger(x'),\ \phi(x)\right]_+\Big|_{x=x'} , \tag{77}$$

where $\hat{P}_\mu = i\frac{\partial}{\partial x^\mu} - q\,A_\mu(x)$ and $\hat{P'}^*_\mu = -i\frac{\partial}{\partial x'^\mu} - q\,A_\mu(x')$, and operator of Chernikov-Tagirov metric energy-momentum tensor (EMT) of the field operator [25] reads

$$T_{\mu\nu} = T^{can}_{\mu\nu} + t_{\mu\nu} , \tag{78}$$

$$T^{can}_{\mu\nu} = B_{\mu\nu}\frac{1}{2}\left[\phi^\dagger(x'),\ \phi(x)\right]_+\Big|_{x=x'} , \tag{79}$$

$$B_{\mu\nu} = \hat{P'}^*_\mu\hat{P}_\nu + \hat{P'}^*_\nu\hat{P}_\mu - \eta_{\mu\nu}\left(\hat{P'}^*_\mu\hat{P}^\mu - M^2\Omega^2(x_0)\right) , \tag{80}$$

$$t_{\mu\nu} = C_{\mu\nu}\frac{1}{2}\left[\phi^\dagger(x),\ \phi(x)\right]_+ , \tag{81}$$

$$C_{\mu\nu} = -\frac{1}{3}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial_\lambda\partial^\lambda\right) , \tag{82}$$

where $T^{can}_{\mu\nu}$ is canonical EMT operator. Here $\phi(x)$ is scalar, j_μ is vector, and $T_{\mu\nu}$ is tensor under proper homogeneous Lorentz transformation. For $d = 4$ one can get operators of scalar field $\Phi(x)$, the current vector J_μ and EMT $\tau_{\mu\nu}$ under general coordinate transformation by using formulas

$$\Phi(x)=\Omega^{-1}(x^0)\phi(x),\quad J_\mu=\Omega^{-2}(x^0)j_\mu,\quad \tau_{\mu\nu}=\Omega^{-2}(x^0)T_{\mu\nu}. \tag{83}$$

We are going to discuss the following matrix elements with these operators

$$<j_\mu>^c=<0,out|j_\mu|0,in>c_v^{-1}\ , \tag{84}$$

$$<T_{\mu\nu}>^c=<0,out|T_{\mu\nu}|0,in>c_v^{-1}\ , \tag{85}$$

$$<j_\mu>^{in}=<0,in|j_\mu|0,in>\ , \tag{86}$$

$$<T_{\mu\nu}>^{in}=<0,in|T_{\mu\nu}|0,in>\ , \tag{87}$$

$$<j_\mu>^{out}=<0,out|j_\mu|0,out>\ , \tag{88}$$

$$<T_{\mu\nu}>^{out}=<0,out|T_{\mu\nu}|0,out>\ \ . \tag{89}$$

Using GF which were found before, one can present these matrix elements in the following form,

$$<j_\mu>^c=q\left(\hat{P}_\mu+\hat{P'}^*_\mu\right)(-i)\Delta^c(x,x')\Big|_{x=x'}\ , \tag{90}$$

$$<T_{\mu\nu}>^c=B_{\mu\nu}(-i)\Delta^c(x,x')|_{x=x'}+C_{\mu\nu}(-i)\Delta^c(x,x)\ , \tag{91}$$

$$<j_\mu>^{in}=<j_\mu>^c+<j_\mu>^{(1)}+<j_\mu>^{(2)}\ , \tag{92}$$

$$<j_\mu>^{out}=<j_\mu>^c+<j_\mu>^{(1)}-<j_\mu>^{(2)}\ , \tag{93}$$

$$<T_{\mu\nu}>^{in}=<T_{\mu\nu}>^c+<T_{\mu\nu}>^{(1)}+<T_{\mu\nu}>^{(2)}\ , \tag{94}$$

$$<T_{\mu\nu}>^{out}=<T_{\mu\nu}>^c+<T_{\mu\nu}>^{(1)}-<T_{\mu\nu}>^{(2)}\ , \tag{95}$$

$$<j_\mu>^{(1,2)}=q\left(\hat{P}_\mu+\hat{P'}^*_\mu\right)(-i)\Delta^{(1,2)}(x,x')\Big|_{x=x'}\ , \tag{96}$$

$$<T_{\mu\nu}>^{(1,2)}=B_{\mu\nu}(-i)\Delta^{(1,2)}(x,x')\Big|_{x=x'}+C_{\mu\nu}(-i)\Delta^{(1,2)}(x,x)\ , \tag{97}$$

where GF are given by eq. (52), (56) and (57), and the relation

$$\Delta^c(x,x)=\frac{1}{2}\left[\Delta^-(x,x)-\Delta^+(x,x)\right]$$

is used.

The components $<j_\mu>^{(1,2)}$ and $<T_{\mu\nu}>^{(1,2)}$ in the expressions (92) and (94) can not be calculated in the frame of the perturbation theory with respect to the external background

or in the frame of WKB method. Among them only the term $< j_\mu >^{in}$ was calculated before and only in the pure electric field in flat space ($b = 0$), see [5]. The only expression (91) for $< T_{\mu\nu} >^c$ has to be regularized and renormalized. The expression (90) for term $< j_\mu >^c$ is finite after the regularization lifting. The terms $< j_\mu >^{(1,2)}$ and $< T_{\mu\nu} >^{(1,2)}$ are also finite. That is consistent with the fact that the ultraviolet divergences have a local nature and result (as in the theory without external field) from the leading local terms at $s \to +0$. The nonzero contributions to the expressions $< j_\mu >^{(1,2)}$ and $< T_{\mu\nu} >^{(1,2)}$ are related to global features of the theory and indicate the vacuum instability.

Let us introduce the normalized values of current and EMT (which maybe easily connected with observable values),

$$j^{cr}_\mu = \tilde{j}^{cr}_\mu / \Omega^3(x_0) \,, \tag{98}$$

$$T^{cr}_{\mu\nu} = \tilde{T}^{cr}_{\mu\nu} / \Omega^3(x_0) \,, \tag{99}$$

(in some convenient asymptotic region $x_0 = x_0^{as}$), where the corresponding densities per space-coordinates volume are

$$\tilde{j}^{cr}_\mu = \frac{\int d\mathbf{x} \, (< j_\mu >^{in} - < j_\mu >^{out})}{\int d\mathbf{x}} \,, \quad x_0 = x_0^{as}, \tag{100}$$

$$\tilde{T}^{cr}_{\mu\nu} = \frac{\int d\mathbf{x} \, (< T_{\mu\nu} >^{in} - < T_{\mu\nu} >^{out})}{\int d\mathbf{x}} \,, \quad x_0 = x_0^{as} \,, \tag{101}$$

according to the definitions (86) - (89).

Then, using representations (92) - (97) one gets from (100) and (101),

$$\tilde{j}^{cr}_\mu = 2 < j_\mu >^{(2)} \,, \quad x_0 = x_0^{as}, \tag{102}$$

$$\tilde{T}^{cr}_{\mu\nu} = 2 < T_{\mu\nu} >^{(2)} \,, \quad x_0 = x_0^{as}. \tag{103}$$

To study the backreaction of particles created on the electromagnetic field and metrics one needs the expressions $< j_\mu >^{in}$ and $< T_{\mu\nu} >^{in}$ at all times x_0. Below we are going to analyse these expressions for different times. Note, that the functions $\Delta^c(x, x')$ (52) and $\Delta^{(1)}(x, x')$ (56) at $x = x'$ are even functions on x_0. Thus, the functions $< T_{\mu\nu} >^c$ and $< T_{\mu\nu} >^{(1)}$ are also even ones and do not vanish at $x_0 \to 0$, whereas the functions $< j_\mu >^c$

and $< j_\mu >^{(1)}$ are odd ones and vanish in this limit. Moreover, we have $< j_\mu >^c= 0$ for all x_0 at $b = 0$.

The proper-time integral $\Delta^{(2)}(x, x')$ (57) is a odd function on x_0 at $x = x'$ and vanishes at $x_0 \to 0$. Thus, the expression $< T_{\mu\nu} >^{(2)}$ is also a odd function on x_0 and vanishes in this limit. The term $< j_\mu >^{(2)}$ is an even function on x_0 and is different from zero in this limit if $E \neq 0$.

One can see, using the expressions obtained for GF, that all off-diagonal matrix elements of EMT are equal to zero:

$$< T_{\mu\nu} >^c=< T_{\mu\nu} >^{(1)}=< T_{\mu\nu} >^{(2)}= 0,\ \mu \neq \nu; \tag{104}$$

and only x^D components (along the electric field) of the currents are different from zero and vanish in the absence of the electric field.

Below we are going to analyse contributions to the physical quantities (102) and (103) for $d = 4$ at $x_0 = x_0^{as}$, which are stipulated by the GF $\Delta^{(2)}(x, x')$.

Using the asymptotic form of $\Delta^{(2)}(x, x')$ given in (A8), one can see that at $x_0 >> \sqrt{\rho}/(bM)$ the following asymptotic expression takes place

$$< j_\mu >^{(2)}= q^2|E|\rho^{-1}\tilde{n}^{cr}\delta^3_\mu. \tag{105}$$

Taking into account eq. (A5) one gets an asymptotic expression for diagonal $< T_{\mu\nu} >^{(2)}$:

$$\begin{aligned}
&< T_{00} >^{(2)}= \left[\rho^2 x_0^2 + a^2M^2 + qH\coth\left(\pi qH/\rho\right)\right]\frac{\tilde{n}^{cr}}{\rho x_0}\ ,\\
&< T_{11} >^{(2)}=< T_{22} >^{(2)}= qH\coth\left(\pi qH/\rho\right)\frac{\tilde{n}^{cr}}{2\rho x_0}\ ,\\
&< T_{33} >^{(2)}= \left[(qEx_0)^2 + \frac{\rho^3}{2\pi(bM)^2}\right]\frac{\tilde{n}^{cr}}{\rho x_0}\ ,
\end{aligned} \tag{106}$$

where $\tilde{n}^{cr}$ is given by (74).

Doubling the expression (105) and (106) according the eq. (102) and (103), one gets the mean densities for current and EMT of particles created. It turns out that these quantities are proportional to the density of total number of particles and antiparticles created $2\tilde{n}^{cr}$

for the infinite time and do not change their structure with increasing of x_0. The latter means that one can consider all the expressions obtained at any fixed x_0 if $x_0 \gg \sqrt{\rho}/bM$. For a strong background $a^2M^2/\rho \leq 1$ and therefore this time has to obey the condition $x_0 >> a/b$. Thus, in the strong background our asymptotic conformal time x_0 corresponds to the large cosmological time t.

Note, that one can neglect the second term in the brackets of the expression (106) for $< T_{33} >^{(2)}$ at $bM/(qE) \leq 1$. Also one can neglect both the term a^2M^2 in the brackets of the expression (106) for $< T_{00} >^{(2)}$ in case of strong external background $a^2M^2/\rho \leq 1$ and third term in the same expression if the magnetic field strength is not big enough $qH/\rho \leq 1$. The current density $\tilde{j}^{cr}_\mu = 2 < j_\mu >^{(2)}$ does not depend of the asymptotic time. At $b \to 0$ this expression coincides with one for flat space $< \tilde{j}_\mu >^{cr} = 2|q|\tilde{n}^{cr}\delta^3_\mu$. The pressure component along the electric field direction $\tilde{T}^{cr}_{33} = 2 < T_{33} >^{(2)}$ is growing with time upon the action of the field. However, if $qE/(bM) << 1$ then the asymptotic condition for x_0 is consistent with the fact that the term $(qEx_0)^2$ in the expressions $< T_{33} >^{(2)}$ from (106) will not be dominant until big enough time x_0. In this case one can neglect the contribution which depends on the electric field if the field is switched off before. Note, that more accurate analysis of back-reaction of created particles to current requires the numerical estimations (compare with purely EM case, [28]).

The components of the pressure in the directions perpendicular to the electric field $\tilde{T}^{cr}_{11} = 2 < T_{11} >^{(2)}$ and $\tilde{T}^{cr}_{22} = 2 < T_{22} >^{(2)}$ are growing in relation to the $\tilde{T}^{cr}_{33}$ when the magnetic field is increasing. However, a very strong magnetic field decreases all the components of $\tilde{T}^{cr}_{\mu\nu}$ and $\tilde{j}^{cr}_\mu$ because the particles creation will be decreased.

One can remark that the asymptotic behaviour of $< j_\mu >^{(1)}$ and $< T_{\mu\nu} >^{(1)}$ is defined by the asymptotic expression for $\Delta^{(1)}(x, x')$ (A11). Then one gets

$$< j_\mu >^{(1)} = < j_\mu >^{(2)}, \quad < T_{\mu\nu} >^{(1)} = < T_{\mu\nu} >^{(2)} \quad . \tag{107}$$

The expression (90) does not need to be renormalized, thus, one can easily verify (using (A1) and (A2)) that the relation $< j_\mu >^c \sim x_0^{-1} \to 0$ holds asymptotically. An estimation of

the finite part of $< T_{\mu\nu} >^c$ can be made only after renormalization. We are going to consider this problem, together with others, related to the renormalization, in the next paper.

To get an idea about the behaviour of the expressions (92) and (94) at finite time let us estimate their components for some small x_0, namely for $x_0^2 << \rho/(bM)^2$. Dislocating the contour Γ_2 to the real axis and neglecting the divergent terms in $< T_{\mu\nu} >^c + < T_{\mu\nu} >^{(1)}$ which do not depend on the background, one can see that

$$< j_\mu >^{in} = \Re < j_\mu >^c + < j_\mu >^{(2)} + < j_\mu >^{(3)} , \tag{108}$$

$$< T_{\mu\nu} >^{in} = \Re < T_{\mu\nu} >^{ren} + < T_{\mu\nu} >^{(2)} + < T_{\mu\nu} >^{(3)} . \tag{109}$$

Here $< T_{\mu\nu} >^{ren}$ is obtained from $< T_{\mu\nu} >^c$ as a result of such a procedure and

$$< j_\mu >^{(3)} = q \left(\hat{P}_\mu + \hat{P'}^*_\mu\right)(-i)\Delta^{(3)}(x,x')\Big|_{x=x'} , \tag{110}$$

$$< T_{\mu\nu} >^{(3)} = B_{\mu\nu}(-i)\Delta^{(3)}(x,x')\Big|_{x=x'} + C_{\mu\nu}(-i)\Delta^{(3)}(x,x) , \tag{111}$$

where $\Delta^{(3)}(x,x')$ is defined in (A9). As was already remarked, $< j_\mu >^c$ and $< j_\mu >^{(3)}$ are odd functions on x_0. They vanish at $x_0 \to 0$. That is why the leading term in (108) is defined by $< j_\mu >^{(2)}$. Using the expressions (B4), obtained in the Appendix B, one gets

$$< j_\mu >^{(2)} = q^2 n^{(2)} \delta^3_\mu . \tag{112}$$

If $bM/(qE) \geq 1$, then the odd function $< T_{\mu\nu} >^{(2)}$ vanishes at $x_0 \to 0$ and the leading contribution to (109) is defined by $\Re < T_{\mu\nu} >^{ren}$ and $< T_{\mu\nu} >^{(3)}$. However, if $bM/(qE) << 1$, in the domain where $qE/(bM)^2 >> x_0^2 >> (qE)^{-1}$, the situation is different and the leading contributions can come from $< T_{\mu\nu} >^{(2)}$. That happens since the expressions $\Re < T_{\mu\nu} >^{ren}$ and $< T_{\mu\nu} >^{(3)}$ remain finite at $b \to 0$. Using (B4) and (B5) one can find expression for diagonal $< T_{\mu\nu} >^{(2)}$,

$$\begin{aligned}
< T_{00} >^{(2)} &= \sum_{l=1,2,3} < T_{ll} >^{(2)} + a^2 M^2 x_0 \pi \left(\frac{bM}{qE}\right)^2 \tilde{n}^{cr} , \\
< T_{11} >^{(2)} &= < T_{22} >^{(2)} = qH \coth(\pi H/E) x_0 \pi \left(\frac{bM}{qE}\right)^2 \tilde{n}^{cr} , \\
< T_{33} >^{(2)} &= 5qEx_0 \tilde{n}^{cr} .
\end{aligned} \tag{113}$$

¿From it one can see that $< T_{33} >^{(2)}$ and $< T_{00} >^{(2)}$ are growing unlimited at $b \to 0$ because of increasing of $\tilde{n}^{cr}$.

The expression (112) at $bM/(qE) << 1$ coincides with the asymptotic one (105). That demonstrates that in the quasi-flat metrics, when the particles are created mainly due to the electric field only, any time far enough $(x_0 + T/2 >> (qE)^{-1/2})$ from the initial time $(x_0^{in} = -T/2)$, can be considered as the asymptotic time. However, that is not true for the EMT $< T_{\mu\nu} >^{(2)}$ and the expression (113) differs essentially from the asymptotic one (106). That happens since the vacuum definition for the particles with big longitudinal momenta $(p_3 \geq qEx_0)$, differs essentially from $|0, out >$ vacuum. Namely, those particles mainly contribute to $< T_{\mu\nu} >^{(2)}$. Technically that means that one can not obtain the normal form by means of the Δ^{out}-function and, consequently, to calculate the mean value of EMT of particles created at $x_0 << \sqrt{qE}/(bM)$ it is not enough to know components of $< T_{\mu\nu} >^{in}$. Note finally that the expressions for mean values of energy momentum tensor may be applied for the estimations of back-reaction of created particles to the gravitational background like it has been done in [26]. However, such study involves the renormalization of EMT. It also cannot be done analytically. We will discuss these questions in an another place.

IV. EFFECTIVE ACTION IN SCALAR QED IN FRW UNIVERSE WITH CONSTANT ELECTROMAGNETIC FIELD

In this section (which is somehow outside of above discussion) we will consider $d = 4$ scalar QED in curved spacetime with EM field. That is more complicated, interacting theory. Using such a theory as an example we will briefly discuss how one can generalize the results of above discussion to interacting theories. We again consider gravitational-electromagnetic background of weakly curved constant curvature spacetime. The Lagrangian of the theory may be written as following:

$$L = L_m + L_{\text{ext}} \ , \tag{114}$$
$$L_m = \frac{1}{2}\left(\partial_\mu\phi_1 - qA_\mu\phi_2\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_2 + qA_\mu\phi_1\right)^2 - \frac{1}{2}M^2\phi^2 + \frac{1}{2}\xi R\phi^2 - \frac{1}{4!}\lambda\phi^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ ,$$

$$L_{\rm ext} = \Lambda + \kappa R + a_1 R^2 + a_2 C^2_{\mu\nu\alpha\beta} + a_3 G \ ,$$

where $\phi^2 = \phi_i\phi_i = \phi_1^2 + \phi_2^2$, the parameterization of fields in terms of two real scalars is taken, R is curvature, $C_{\mu\nu\alpha\beta}$ is Weyl tensor, G is Gauss-Bonnet combination. The necessity of introduction of $L_{\rm ext}$ is dictated by the multiplicative renormalizability of the theory (see [12]). Note also that we consider the theory in curved spacetime with EM field, hence $A_\mu \to A_\mu + \tilde{A}_\mu$, where $\tilde{A}_\mu$ is background EM field. For simplicity, we limit below only in constant curvature space. The well known example of such spaces is De Sitter space which is often used for description of the inflationary Universe (exponentially expanding one).

There are few ways to study such a theory on the quantum level. When one treats the external background exactly, like it has been done in previous section, the first step is to calculate scalar Green functions.

Due to the fact that the Maxwell theory is conformally invariant, Green functions for EM field will be the same as in the flat space. Then, using the proper-time representation for the Green functions, quantum corrections to $\Gamma_{\rm out-in}$ or $\Gamma_{\rm in-in}$ can be calculated (where $\Gamma_{\rm out-in}$, $\Gamma_{\rm in-in}$ are effective actions for the probability amplitudes and for the mean values respectively). Then, of course, the external background is kept to be exact and perturbation theory over only coupling constants λ, e is used. However, such calculation is extremely hard what can be already understood from very complicated form of the Green functions in the previous section. Indeed, these Green functions should be used instead of standard momentum space propagators in Feynman diagrams (for an explicit example in pure constant EM background, see [5]). Hence, such calculation using Green functions of the previous section will be given in another place.

Instead, we will adopt quasi-local expansion for the effective action here. We will make the explicit use of RG [12] in such calculation. Note that in quasi-local approach to effective action (i.e. when we do not treat an external background exactly but make the derivative expansion of the effective action over derivatives from metric and EM field) the difference between $\Gamma_{\rm out-in}$, and $\Gamma_{\rm in-in}$ is not seen. That is why we denote the effective action here

as Γ simply. (Of course, particle creation phenomenon is also hard to see in quasi-local approximation.) As background we will consider De Sitter space with background EM field A_μ (no background scalar field!).

Using the fact that the theory with the Lagrangian (115) is multiplicatively renormalizable, one can write RG equation for the effective action on the above background as following:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_i\frac{\partial}{\partial\lambda_i}-\gamma_A\tilde{A}_\mu\frac{\partial}{\partial\tilde{A}_\mu}\right)\Gamma(\mu,\lambda_i,\tilde{A}_\mu,g_{\mu\nu})=0\ , \tag{115}$$

where $\tilde{A}_\mu$ is background EM field chosen so that $\tilde{F}_{\mu\nu}$ is (almost) constant (the well-known example is constant electric field), $\lambda_i=(q,\lambda,M,\xi,\Lambda,\kappa,a_1,a_2,a_3)$, γ_A is γ-function for $\tilde{A}_\mu$, and β_i is β-function for λ_i.

The solution of RG equation (115) can be easily found using the method of characteristics (see [15] for flat space and [16] for curved spacetime). In such a way, one obtains RG improved effective action (which makes summation of all leading logarithms of the perturbation theory). Dropping the details which are very similar to the ones given in ref. [16], we get RG improved effective action up to the terms of second order on the curvature invariants:

$$\Gamma_{RG}=\Lambda(t)+\kappa(t)+a_1(t)R^2+a_3(t)G-\frac{1}{4}\frac{q^2}{q^2(t)}\tilde{F}^2_{\mu\nu}\ , \tag{116}$$

where the effective coupling constants are given as following (see [15] for the flat space and [16] for curved spacetime):

$$\begin{aligned}
&q^2(t)=q^2\left(1-\frac{2q^2t}{3(4\pi)^2}\right)^{-1}, \quad \Phi^2(t)=\Phi^2\left(1-\frac{2q^2t}{3(4\pi)^2}\right)^{-9},\\
&\lambda(t)=\frac{1}{10}q^2(t)\left[\sqrt{719}\tan\left(\frac{1}{2}\sqrt{719}\log q^2(t)+C\right)+19\right],\\
&C=\arctan\left[\frac{1}{\sqrt{719}}\left(\frac{10\lambda}{q^2}-19\right)\right]-\frac{1}{2}\sqrt{719}\log q^2,\\
&M^2(t)=M^2\left[\frac{q^2(t)}{q^2}\right]^{-26/5}\frac{\cos^{2/5}\left(\frac{1}{2}\sqrt{719}\log q^2+C\right)}{\cos^{2/5}\left(\frac{1}{2}\sqrt{719}\log q^2(t)+C\right)},\\
&\xi(t)=\frac{1}{6}+\left(\xi-\frac{1}{6}\right)\left[\frac{q^2(t)}{q^2}\right]^{-26/5}\frac{\cos^{2/5}\left(\frac{1}{2}\sqrt{719}\log q^2+C\right)}{\cos^{2/5}\left(\frac{1}{2}\sqrt{719}\log q^2(t)+C\right)},
\end{aligned}$$

$$
\begin{aligned}
&a_2(t) = a_2 + \frac{7t}{60(4\pi)^2}, \quad a_3(t) = a_3 - \frac{8t}{45(4\pi)^2}, \\
&\lambda(t) = \Lambda + M^4 A_1(t), \quad \kappa(t) = \kappa + 2M^2(\xi - 1/6)A_1(t), \quad a_1(t) = a_1 + (\xi - 1/6)^2 A_1(t), \\
&A_1(t) = \int_0^t \frac{dt}{(4\pi)^2} \left[\frac{q^2(t)}{q^2}\right]^{-52/5} \frac{\cos^{4/5}\left(\frac{1}{2}\sqrt{719}\, \log q^2 + C\right)}{\cos^{4/5}\left(\frac{1}{2}\sqrt{719}\, \log q^2(t) + C\right)} . \qquad (117)
\end{aligned}
$$

Note that one should use the fact that $C_{\mu\nu\alpha\beta} = 0,\ G = \frac{R^2}{6}$ on de Sitter background. The classical Lagrangian is used as boundary condition in the derivation of Γ_{RG}.

Now, the question is about the choice of RG parameter t. We are dealing with the theory with few effective masses. RG improvement in such a theory is not easy due to the fact that one has generally speaking to generalize the total mass matrix (there are two masses from the scalar sector plus four more from EM sector, see [16,17]).

In order to find RG parameters t one has to specify situation in more details. For example, let us consider the case when the background EM field is chosen to be constant magnetic field, i.e. $\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}H^2$. Then, one can consider different limiting cases.

First, let $H \gg R$. Then all effective masses are becoming proportional and there is unique choice for RG parameter $t : t = \frac{1}{2}\ln\frac{qH}{\mu^2}$. In this case, the curvature effects are not relevant, Γ_{RG} is given by its last term in (117). That is the case actually discussed by Schwinger [1].

Second, let $R \gg H$. Then, the unique choice for t is $t = \frac{1}{2}\ln\frac{R/4}{\mu^2}$. Now, EM field effects are not relevant. We get purely gravitational effective action discussed in ref. [12,13,16].

When both fields are of the same order the choice for t (in order to make summation of leading logarithms) is $t = \frac{1}{2}\ln\frac{R/4+qH}{\mu^2}$. If, in addition, we choose the initial values for mass and ξ as following: $M^2 = 0,\ \xi = \frac{1}{6}$, then we get

$$
\Gamma_{RG} = -\frac{8t}{270(4\pi)^2}R^2 - \left(1 - \frac{2q^2 t}{3(4\pi)^2}\right)\frac{H^2}{2}\ , \qquad (118)
$$

where only H-dependent terms kept. The expression (118) generalizes the well-known effective Lagrangian for magnetic field obtained by Schwinger [1] to curved spacetime. Note that in (118) the curvature effects have been taken into account.

It is known that in flat space [1] the Schwinger effective Lagrangian leads to the stationary point $H \neq 0$ (due to quantum corrections). However, this stationary point is known to be the maximum of the effective action. Hence, it does not lead to the possibility of new ground state with account of quantum corrections. In curved space, one can analyze Γ_{RG} (118) using equation of motion $\frac{\partial \Gamma_{RG}}{\partial H}$. The result of this analysis shows that position of the flat-space maximum $H \neq 0$ is moving due to curvature corrections. However, the nature of this stationary point is the same (it is the maximum of the effective action). Similarly, one can analyze (now numerically) the case with $M^2 \neq 0$ and an arbitrary ξ. However, we expect that existence of new ground state may occur only in the regime with strong curvature (where an external gravitational field is treated exactly like in the previous section).

V. CONCLUSION

In summary, we considered massive scalar field in expanding radiation dominated FRW Universe filled by the constant EM field. Taking scalar-gravitational coupling constant to be equal to its conformal value and making conformal transformation, the theory is formulated as flat space theory with time-dependent mass and in external EM field. For such a theory, which is generally speaking theory with unstable vacuum, we found proper-time representation for all Green functions (i.e. for in-in, out-in, and so on Green functions). Note that proper-time representation (over a finite contour in proper-time complex plane) for in-in Green function in the combination of the external gravitational and EM fields is given here for the first time.

As some applications, we discuss particles creation by the external fields in arbitrary dimensions. Combined action of gravitational and EM fields may produce new interesting features in this phenomenon, in particular, affect the rate of particles creation.

Using the proper-time representation for Green functions, the proper-time representation for the effective actions is also found (or, in other words, vacuum polarization is found) keeping external background exactly.

Let us discuss one more possible application of the Green functions obtained-the calculation of radiative corrections to the transition amplitudes and to mean values of different quantities in the interacting theory. The corresponding general theory (Furry picture for theories with unstable vacuum) was developed in [3–6]. As an example, let us consider scalar self-interacting theory with the interaction $-\frac{1}{4!}(\phi^+\phi)^2$. The same background as in Sect. II will be considered. Let us imagine that one has to calculate first-order radiative corrections to the two-point Green function which serves for calculation of mean values. Schematically, we have to consider the following diagram;

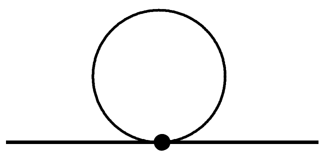

FIG. 4. A radiative correction to the propagator

with Δ_{in} functions. Hence, these corrections are proportional to in-in propagator in coinciding points, i.e. $\sim \Delta^c_{in}(x,x)$. Or more completely, above-drawn diagram is proportional to the expression

$$\sim \int \Delta(y,x')\Delta(x',x')\Delta(x',z)dx' \ .$$

The divergent part of the diagram is well-known, it is the same as in flat space and is proportional (in dimensional regularization) to $\frac{\lambda M^2}{(n-4)}$. The finite part of the diagram can be easily found, using explicit form of the proper-time representation for the corresponding Green function, given in Sect. II. Of course, this finite part is different from one in the absence of the external background. To calculate the finite radiative corrections to the causal propagator, one has to take the same diagram with Δ^c functions (or with out-in Green functions). The divergent part of the diagram will be the same, however, the finite part is different from the previous case (as it follows from the explicit structure of Green functions, presented in Sect. II). In the same way one can analyse other radiative corrections.

Using explicit form of Green functions in Sect. II, we calculate the effective actions (vacuum polarization) in proper-time representation. The knowledge of out-in effective

action gives an alternative way to define the particles creation a la' Schwinger [1] (for an explicit example see [12]). The in-in effective action can be used to study the back reaction of the particles created to the external background. Such an analysis is not easy and will be presented in another place where also a generalization of results of Sect. II for an arbitrary ξ will be done. For example, using explicit form of in-in GF in proper-time representation it could be of interest also to construct proper-time representation for in-in effective action.

In Sect. IV we, presented another approach to the effective action (derivative expansion of effective action) in the external gravitational-EM background. Scalar QED is considered as an example; RG improved effective action (up to the terms of second order on curvature and EM strength) is calculated on constant curvature weakly curved spacetime with weak constant EM field. Such an effective action gives the extension of the well-known Schwinger effective Lagrangian, taking into account curvature effects. It may be also applied to the study of back reaction of quantum field theory to external background.

Finally, similar technique may be applied to analyze the behaviour of spinor fields in gravitational-EM background. The calculation of all the Green functions in such a theory, using proper-time representation, may be the necessary step in the study of chiral symmetry breaking in QED and in the four-fermion models under the action of gravitational and EM fields. Such a study may have an immediate important application to early Universe, for example, through the construction of inflationary Universe where role of inflaton is played by the condensate $< \bar{\Psi}\Psi >$. One can also analyse symmetry breaking phenomenon under the combined action of gravitational and EM fields in the Standard Model (using also its gauged NJL form [27]), or Grand Unified Theories in the same way as it has been done in curved spacetime (without EM field) [12].

Note also that GF investigation developed in this paper maybe extremly useful for the study of Casimir effect due to combined action of gravitational and electromagnetic fields (for an introduction to Casimir effect in pure EM or pure gravitational case, see [30]).

VI. ACKNOWLEDGMENTS

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APPENDIX A: ASYMPTOTICS OF Δ- FUNCTIONS

Let us calculate the asymptotic behavior of $\Delta^{(1)}(x,x')$ and $\Delta^{(2)}(x,x')$ in the case $x_0^2 >> \rho/(bM)^2$, and $x \to x'$ at $d=4$.

First, it is useful to take into account the following formulas

$$\hat{P}_\mu e^{iq\Lambda} = e^{iq\Lambda} i\partial_\mu \ , \quad \hat{P'}^*_\mu e^{iq\Lambda} = e^{iq\Lambda}(-i\partial'_\mu) \ , \tag{A1}$$

$$\hat{P}_j f(x,x',s) = \hat{P'}^*_j f(x,x',s) \ , \quad j=1,2,3 \ , \tag{A2}$$

$$\hat{P}_0 \hat{P'}^*_0 f(x,x's)\Big|_{x=x'} = \left[\hat{P}_0^2 + 2\left(\omega^{-1}(bM)^2 s x_0\right)^2 - i\omega^{-1}(bM)^2 s\right] f(x,x',s)\Big|_{x=x'} \ . \tag{A3}$$

Then, using eq.(A3) and equation (62) one gets

$$\hat{P}_0 \hat{P'}^*_0 f(x,x',s)\Big|_{x=x'} = \\ \left[-i\frac{d}{ds} + \hat{P}_j^2 + M^2\Omega^2(x_0) + 2\left(\omega^{-1}(bM)^2 s x_0\right)^2 - \omega^{-1} i (bM)^2 s\right] f(x,x',s)\Big|_{x=x'} \ . \tag{A4}$$

If the Δ-function obeys the equation (2) then the action of the operator $-i\frac{d}{ds}$ on this function is equal to zero. Further we are going to use the formula (A4) for simplification of the calculation. Besides, one can see that for such kind of functions the following relation holds,

$$\left(\hat{P'}^*_\mu \hat{P}^\mu - M^2\Omega^2(x_0)\right)\Delta^{(\dots)}(x,x')\Big|_{x=x'} = 2\partial_0^2 \Delta^{(\dots)}(x,x) \ . \tag{A5}$$

So, if one uses these formulas in calculations it is enough to find GF asymptotics for the case $x_0 - x'_0 = 0$.

Let us find the asymptotic behavior of $\Delta^{(2)}(x,x')$ function given by the eq. (57). If $b \neq 0$ and $x_0 - x'_0 = 0$ the kernel $f_r(x,x's)$ has no singular point $s_1 = -i\pi/\rho$. Below the line of

contours $\Gamma_3 - \Gamma_a$ next singular point of this function is $s_2 - i\pi/\rho$. Thus, one can shift the line of the composite contour $\Gamma_3 - \Gamma_a$ below along the imaginary axis until the neighborhood of the point $s_2 - i\pi/\rho$. The contour which is obtained in such a way from $\Gamma_3 + \Gamma_2 - \Gamma_a$ can be closed on the parts $\Re s \to \pm\infty$ and then can be transformed into a closed contour which includes the points s_1 and s_2 as well. At the same time this contour can be situated far enough from the point s_1, so that $|\rho s \omega^{-1}|$ is always not zero at it. Then one can use the asymptotic decomposition [23]

$$\gamma\left(1/2, \alpha\right) = \sqrt{\pi} - e^{-\alpha}\alpha^{-1/2}\left[1 + O(\alpha^{-1})\right] \ , \ \ x_0 > 0 \ . \tag{A6}$$

The contribution from the first term of eq. (A6) at the contour in question is exponentially small since $\Re(-i\rho s\omega^{-1}) < 0$. The rest terms of eq. (A6) form a series in inverse powers of $x_0^2(bM)^2/\rho$, the corresponding functions in the decomposition coefficients $f(x,x',s)e^{-\alpha}\alpha^{-1/2}\left[1 + O(\alpha^{-1})\right]$ have only one singular point s_1 (the pole) inside the contour. The contributions from these terms can be estimated tightening the contour to the point s_1. The first one of these terms defines the leading contribution into the asymptotics of $\Delta^{(2)}(x,x')$,

$$\Delta^{(2)}(x,x') = \frac{1}{2\sqrt{\pi}}\int_{\Gamma_R^1} f(x,x',s)e^{-\alpha}\alpha^{-1/2}ds \quad , \tag{A7}$$

where the contour Γ_R^1 (see FIG.3) is a circle with infinitesimal radius around the singular point s_1. Calculating the residue, one gets an expression for $\Delta^{(2)}(x,x')$ which defines the leading asymptotics in $< j_\mu >^{(2)}$ and $< T_{\mu\nu} >^{(2)}$,

$$\Delta^{(2)}(x,x') = i\frac{\tilde{n}^{cr}}{\rho(x_0+x_0')}\exp\{iq\Lambda + \frac{i}{2}qE(x_0+x_0')y^3 - \frac{\rho^3(y_3)^2}{4\pi(bM)^2} + \frac{1}{4}y_\perp qF\cot\left(\pi qF/\rho\right)y_\perp\} \quad , \tag{A8}$$

where $\tilde{n}^{cr}$ is defined in (74). This expression is also valid at $x_0 < 0$, since the function $\Delta^{(2)}(x,x')$ is an odd one in x_0 at $x = x'$.

Consider the asymptotic behavior of $\Delta^{(1)}(x,x')$ function given by eq. (56). Since $\rho s\omega^{-1}$ is not equal to zero and $\Re(-i\rho s\omega^{-1}) < 0$ at the contour Γ_2 the corresponding asymptotic

contribution in integral (56) is exponentially small. Then one needs to evaluate only the part of this integral given by form

$$\Delta^{(3)}(x,x') = -\frac{1}{2}\int_{\Gamma_3+\Gamma_a} f(x,x',s)ds \quad , \tag{A9}$$

Let us introduce the variable τ, $\rho s = -i\pi + \tau$. Since $\Re(-i\rho s\omega^{-1}) < 0$ at the contours Γ_3 and Γ_a, and since $|\omega|^{-1}$ increases monotonous with $|\tau|$, then the leading asymptotic contribution is defined by the behaviour of the function $f(x,x',s)$ at small τ and has the form

$$\Delta^{(3)}(x,x') = e^{-i\pi/4}\pi^{-1/2} f(x,x',s_1)\int_0^\infty d\tau\tau^{-1}e^{-i\tau\rho x_0^2} \quad . \tag{A10}$$

Calculating the integral one gets

$$\Delta^{(1)}(x,x') = \mathrm{sign}(x_0)\Delta^{(2)}(x,x') \quad , \tag{A11}$$

where $\Delta^{(2)}(x,x')$ is given eq.(A8).

APPENDIX B: SMALL TIME EXPANSION OF $\Delta^{(2)}$-FUNCTION

Let calculate a small time expansion of $\Delta^{(2)}(x,x')$ function given by eq. (57) in a case $x_0^2 << \rho/(bM)^2$ and $x \to x'$ at $d = 4$. The small α expansion of the incomplete γ-function is valid at the contours Γ_a, Γ_2 and Γ_3 and it has a form [23]

$$\gamma\left(1/2,\alpha\right) = e^{-\alpha}\alpha^{1/2}\left[2 + (4/3)\alpha + 0\left(\alpha^2\right)\right] \quad , \tag{B1}$$

where second term in the square brackets is necessary for the calculations of $\hat{P'}^*_0\hat{P}_0$ and $\hat{P'}^*_3\hat{P}_3$ actions. Since term (68) is equal to zero it is convenient to calculate the function $\Delta^{(2)}(x,x')$, closing the contour $\Gamma_3+\Gamma_2-\Gamma_a$ on the area $\Re s \to \pm\infty$ and then tightening it to the singular point s_2. Then the leading contributions to $< j_\mu >^{(2)}$ and $< T_{\mu\nu} >^{(2)}$, are defined by the integral

$$\Delta^{(2)}(x,x') = \frac{1}{2\sqrt{\pi}}\int_{\Gamma_l+\Gamma_r} f(x,x',s)e^{-\alpha}\alpha^{1/2}\left[2 + (4/3)\alpha\right] ds \quad , \tag{B2}$$

where the contour $\Gamma_l + \Gamma_r$ (see FIG.3) is a infinitesimal radius clockwise circle around the singular point s_2. According to eq. (64) $\omega = 0$ at $s = s_2$. Then in the neighborhood of this singular point if $s = s_2 + s'$ one gets expansion

$$\omega = \omega' \rho s' + (1/2)\omega''(\rho s')^2 \ , \tag{B3}$$
$$\omega' = -i\frac{1}{c_2}\left(\frac{bM}{\rho}\right)^2 \left[c_2^2 + \left(\frac{qE}{bM}\right)^2 + \left(\frac{qE}{bM}\right)^4\right] \ ,$$
$$\omega'' = 2\frac{1}{c_2^2}\left(\frac{bM}{\rho}\right)^2 \left[1 + \left(\frac{qE}{bM}\right)^2\right]\left[c_2^2 + \left(\frac{qE}{bM}\right)^4\right] \ .$$

Calculating residue at the point s_2 one finds the small time expansion of $\Delta^{(2)}$-function,

$$\begin{aligned}
&\Delta^{(2)}(x,x') = e^{iq\Lambda}\left\{\left[i(x_0+x_0')c_2(bM)^2/\rho - qEy^3\right] n^{(2)}/(qE)\varphi_0\right.\\
&\left. + i\left[-(x_0+x_0')^3\frac{(bM)^4}{6qE\rho^2}K(2) + (1/2)(x_0+x_0')(y_3)^2 qEK(0)\right]\right\} \\
&\varphi_0 = \exp\left(i\frac{qE}{2}(x_0+x_0')y^3 - \frac{\rho}{4c_2}\left(\frac{qE}{bM}\right)^2(x_0-x_0')^2 - \frac{\rho^3(y_3)^2}{4c_2(bM)^2}\right) \ , \\
&K(l) = -\frac{(-i)^l c_2^l}{c_2^2 + \left(\frac{qE}{bM}\right)^2 + \left(\frac{qE}{bM}\right)^4}\left\{a^2M^2/\rho - c_2^{-1}(l-1/2) - c_2^{-1}\left(\frac{qE}{bM}\right)^2\right. \\
&\left. +(qH/\rho)\coth(c_2qH/\rho) + 2c_2^{-1}\frac{\left[1+\left(\frac{qE}{bM}\right)^2\right]\left[c_2^2+\left(\frac{qE}{bM}\right)^4\right]}{c_2^2+\left(\frac{qE}{bM}\right)^2+\left(\frac{qE}{bM}\right)^4}\right\}\frac{\rho}{2qE}n^{(2)} \ , \\
&n^{(2)} = \frac{\sqrt{c_2^2+\left(\frac{qE}{bM}\right)^4}}{8\pi^{3/2}c_2\left[c_2^2+\left(\frac{qE}{bM}\right)^2+\left(\frac{qE}{bM}\right)^4\right]}\frac{q^3HE^2\rho^{3/2}}{(bM)^3\sin(c_2qH/\rho)}e^{-c_2a^2M^2/\rho} \ ,
\end{aligned} \tag{B4}$$

If $bM/qE << 1$, coefficients $K(l)$ and $n^{(2)}$ from (B4) have more simple form, and in the case of an intensive electric field ($a^2M^2/(qE) < 1, \ |H/E| < 1$) one has

$$n^{(2)} = \tilde{n}^{cr} \ ,$$
$$K(l) = -(-i)^l(1/2)\pi^l\tilde{n}^{cr} \ . \tag{B5}$$

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